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NEOCLASSICAL TRANSPORT CALCULATIONS FOR OPTIMIZATION STUDIES*

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Abstract: Neoclassical transport coefficients and equilibrium currents are numerically computed using various kinetic equation solvers. In the case when many iterations of this computations are needed, e.g. in optimization problems, fast and simple methods are very interesting. To meet this requirements, new formulas for neoclassical transport coefficients and equilibrium currents have been obtained. These formulas hold in any coordinate system and no simplifying assumptions about the magnetic field are needed. The formulas can be used also for complex magnetic fields which, for actual field coils, are sometimes only available in real space coordinates. It has to be mentioned that the numerical effort for such a numerical computation is rather small, since all the results can be gathered during the magnetic field line integration. Formulas are now available for transport in the $1/\nu$ regime, in the plateau regime and for equilibrium currents (bootstrap and Pfirsch-Schlüter). The coefficients are always expressed in terms of weighted integrals of the geodesic curvature along the magnetic field line. Up to now, there exist two versions of the numerical realization, one version of the code in real space coordinates and one in Boozer coordinates.

1. Introduction

In the present paper, using an analytic solution of the linearized drift kinetic equation in the long-mean-free-path regime, formulas for neoclassical transport coefficients and for the parallel current density are obtained for stellarator configurations with realistic magnetic field geometry. As in standard neoclassical theory, for a solution of the linearized drift kinetic equation, the deviation of the distribution function from a Maxwellian is expanded into a series with respect to the collision frequency. The leading order term in this expansion is proportional to $1/\nu$. This leading term is sufficient to obtain the particle and energy fluxes in this regime. In [1], this term is calculated taking into account all classes of trapped particles. Finally, the results are presented in a form containing a line integral along the magnetic field line and an integration over the perpendicular adiabatic invariant of trapped particles.

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For the calculation of the parallel current density, also the next term in the expansion over the collision frequency is necessary. In contrast to $1/\nu$ transport, where the contribution of multiply trapped particles within many local magnetic field minima is small, they play an essential role in the formation of the parallel current density. In [2], a method to calculate the bootstrap current is proposed which utilizes Boozer coordinates and which is also based on a line integration along the magnetic field line. Here, this procedure is generalized in two ways: (i) the contribution of trapped particles is taken into account; and (ii) calculations can also be done directly in real space coordinates.

The plateau regime of transport in stellarators corresponds to the so called intermediate region of particle collision frequencies for which the effects related to the $1/\nu$ transport are yet not important and for which transport coefficients depend only weakly on the collision frequency. Here, a new formula for the transport coefficients is obtained which is based upon a solution of the drift kinetic equation. Again, also this formula holds for conditions corresponding to the plateau regime without any further restriction for the magnetic field geometry but the existence of intact flux surfaces.

2. Transport and currents in the long mean-free-path regime

The starting point is the linearized drift-kinetic equation in the long-mean-free-path regime with a Lorentz collision operator which describes pitch angle scattering but does not conserve momentum,

$$\sigma \frac{\partial \tilde{f}}{\partial s} + \frac{V^\psi}{|v_\parallel|}\frac{\partial f_M}{\partial \psi} = 4\nu A \frac{\partial}{\partial J_\perp}\left(\frac{|v_\parallel| J_\perp}{B}\frac{\partial \tilde{f}}{\partial J_\perp}\right), \tag{1}$$

where σ is the sign of the parallel velocity, s is the distance measured along the magnetic field line, ψ is the magnetic surface label, $V^\psi = \mathbf{V}\cdot\nabla\psi$ is a radial component of the drift velocity, $v_\parallel^2 = v^2 - J_\perp B$, $J_\perp = v_\perp^2/B$ is the perpendicular adiabatic invariant, B is the magnetic field module, $f_M = f_M(\psi, w)$ is the Maxwellian distribution function, $\tilde{f}$ is the correction to the distribution function according to $f = f_M + \tilde{f}$, $w = mv^2/2 + e\Phi$ is the total energy, Φ is the electrostatic potential, and νA is the pitch-angle scattering frequency. As discussed in [2], for small magnetic field modulations within the magnetic surface, the momentum preserving term will change the resulting value of the average parallel current by a factor which is weakly dependent on the magnetic field geometry and, therefore, can be taken from tokamak theory.

The problem in the interpretation of the boundary condition at the trapped-passing boundary in [2] where the "last" class of trapped particles cannot be identified, does not appear if instead of an irrational surface one first considers a rational surface. In this case, the number of classes of trapped particles stays finite, and the boundary conditions are clearly defined. Then, the irrational surface can be considered as a limiting case of a "true" rational surface [3] which satisfies the closure condition for the equilibrium currents (Pfirsch-Schlüter),

$$Y_{PS}(L) = 0, \qquad Y_{PS}(s) = \int\limits_0^s \mathrm{d}s' \frac{|\nabla\psi| k_G}{B^2}, \tag{2}$$

where

$$k_G = \nabla\psi \cdot \mathbf{h} \times \nabla \ln B / |\nabla\psi| = (\mathbf{h} \times (\mathbf{h} \cdot \nabla)\mathbf{h}) \cdot \nabla\psi / |\nabla\psi|, \tag{3}$$

is the geodesic curvature of the magnetic field line and $\mathbf{h} = \mathbf{B}/B$. With the help of (3), the radial drift velocity can be presented in the form

$$V^\psi \equiv \mathbf{V} \cdot \nabla\psi = (v^2 + v_\parallel^2) \frac{|\nabla\psi| k_G}{2\omega_c} = -|v_\parallel| \frac{\partial}{\partial J_\perp} \frac{|v_\parallel|}{B} V_G, \tag{4}$$

where

$$V_G = (v^2 + \frac{1}{3} v_\parallel^2) \frac{|\nabla\psi| k_G}{\omega_c}. \tag{5}$$

The correction $\tilde{f}$ to the distribution function is expanded into a series over the collision frequency

$$\tilde{f} = f_{-1} + g_0 + f_0 + g_1 + f_1 + \ldots, \tag{6}$$

with $f_k, g_k \sim \nu^k$, where f_k is constant and g_k varies along the magnetic field line. Thus, the kinetic equation is split into a series of equations. Fulfilling a variety of requirements (periodicity, single-valuedness, constraint conditions, boundary conditions), these equations can be solved, yielding

$$\begin{aligned}
\frac{\partial f_{-1}}{\partial J_\perp} &= -\frac{H}{12 \nu A J_\perp I} \frac{\partial f_M}{\partial \psi}, && (7)\\
g_0(s) &= \sigma \frac{\partial f_M}{\partial \psi} \frac{\partial}{\partial J_\perp} \int\limits_{s_{\min}}^{s} \mathrm{d}s' \frac{|v_\parallel|}{B} \left(V_G - \frac{H}{3I} \right), && (8)\\
\frac{\partial f_0}{\partial J_\perp} &= -\frac{\sigma}{I} \frac{\partial f_M}{\partial \psi} \int\limits_{s_{\min}}^{s_{\min}+L} \mathrm{d}s \frac{|v_\parallel|}{B} \frac{\partial^2}{\partial J_\perp^2} \int\limits_{s_{\min}}^{s} \mathrm{d}s' \frac{|v_\parallel|}{B} \left(V_G - \frac{H}{3I} \right), && (9)
\end{aligned}$$

where $s_{\min}$ is the left reflection point for trapped particles or the point of the global magnetic field maximum on the field line for passing particles, and

$$I = \oint \frac{\mathrm{d}s}{B} v_\parallel, \qquad H = 3 \oint \mathrm{d}s \frac{v_\parallel}{B} V_G. \tag{10}$$

3. Particle and energy fluxes in the $1/\nu$ regime

The expression (7) for f_{-1} yields particle and energy fluxes in the $1/\nu$ regime [1]. The definition of the averaged particle flux density, F_n, is given as

$$F_n = \frac{\pi}{mS} \oint\limits_\psi \frac{\mathrm{d}S}{|\nabla\psi|} B \sum_{\sigma=\pm 1} \int \mathrm{d}w \int \mathrm{d}J_\perp f_{-1} \frac{\mathbf{V} \cdot \nabla\psi}{|v_\parallel|} . \tag{11}$$

The equivalence of surface and field line averaging

$$\frac{\langle P \rangle}{\langle Q \rangle} = \oint\limits_\psi \frac{\mathrm{d}S}{|\nabla\psi|} P \left(\oint\limits_\psi \frac{\mathrm{d}S}{|\nabla\psi|} Q \right)^{-1} = \lim_{L_s \to \infty} \int\limits_0^{L_s} \frac{\mathrm{d}s}{B} P \left(\int\limits_0^{L_s} \frac{\mathrm{d}s}{B} Q \right)^{-1} , \tag{12}$$

integration by parts over $J_\perp$, changing the order of integration so that the integral over s becomes the innermost one, integration over energy and reordering of terms, yields

$$F_n = -\frac{\sqrt{8}}{9\pi^{3/2}} \frac{v_T^2 \rho_L^2}{\nu R^2} \epsilon_{\rm eff}^{3/2} \int\limits_0^\infty \frac{{\rm d}z\, e^{-z} z^{5/2}}{A(z)} \frac{n}{f_M} \frac{\partial f_M}{\partial r} \,, \tag{13}$$

where $w = zT + e\Phi$. The quantity $\epsilon_{\rm eff}$ is the so-called effective ripple which replaces the helical ripple, $\epsilon_{\rm h}$, which would present in (13) in case of a standard stellarator, making it applicable for an arbitrary magnetic configuration.

Here, ψ is the magnetic surface label, and R is the major radius of the torus. The derivative of f_M is taken with respect to a formal radius of the magnetic surface, r, which corresponds to the definition

$$\frac{\partial f_M}{\partial r} = \frac{\partial f_M}{\partial \psi} \langle |\nabla\psi| \rangle, \qquad \langle |\nabla\psi| \rangle = \lim_{L_s\to\infty} \left(\int\limits_0^{L_s} \frac{ds}{B} \right)^{-1} \int\limits_0^{L_s} \frac{ds}{B} |\nabla\psi|. \tag{14}$$

For convenience in the further discussion $\epsilon_{\rm eff}$ is presented as

$$\epsilon_{\rm eff}^{3/2} = \hat{\epsilon}_{\rm eff}^{3/2} / \langle |\nabla\psi| \rangle^2 \,, \tag{15}$$

with

$$\hat{\epsilon}_{\rm eff}^{3/2} = \frac{\pi R^2}{8\sqrt{2}} \lim_{L_s\to\infty} \left(\int\limits_0^{L_s} \frac{ds}{B} \right)^{-1} \int\limits_{B_{min}^{(abs)}/B_0}^{B_{max}^{(abs)}/B_0} {\rm d}b' \sum_{j=1}^{j_{max}} \frac{\hat{H}_j^2}{\hat{I}_j} \,, \tag{16}$$

and the additional integrals over s

$$\hat{H}_j = \frac{1}{b'} \int\limits_{s_j^{(min)}}^{s_j^{(max)}} \frac{{\rm d}s}{B} \sqrt{b' - \frac{B}{B_0}} \left(4\frac{B_0}{B} - \frac{1}{b'} \right) |\nabla\psi| k_G, \quad \hat{I}_j = \int\limits_{s_j^{(min)}}^{s_j^{(max)}} \frac{{\rm d}s}{B} \sqrt{1 - \frac{B}{B_0 b'}} \,. \tag{17}$$

The averaged energy flux F_T can be obtained from (13) by multiplying the sub-integrand by the factor zT. For a numerical realization, the integral over the perpendicular adiabatic invariant $J_\perp$ (by means of the variable b') is solved by summation over a given number of particles, whereas the integrals over s are computed by solving simultaneously 1$^{\rm st}$-order differential equations with the Runge-Kutta-method. Here contributions to the integral are switched on and off at the respective boundaries $s_j^{(min)}$ and $s_j^{(max)}$. The quantities $s_j^{(min)}$ and $s_j^{(max)}$ within the sum over j in (17) correspond to the turning points of trapped particles.

For a magnetic field originally available in real space coordinates, there is no necessity in a field transformation to magnetic coordinates. In this case, formulas (15) - (17) must be supplemented with the magnetic field line equations as well as with equations for the vector $\mathbf{P} \equiv \nabla\psi$ (see Ref. [4])

$$\frac{dP_i}{ds} = -\frac{1}{B} \frac{\partial B^j}{\partial \xi^i} P_j \,, \tag{18}$$

where B^j are the contravariant components of $\mathbf{B}$ in real space coordinates ξ^i, and $P_j = \partial\psi/\partial\xi^j$ are the covariant components of $\mathbf{P}$.

Presentation in Boozer coordinates

To deal with transport processes in configurations with finite beta equilibria, Boozer magnetic coordinates are often used. This is also the method of choice in all optimization studies.

The contravariant and covariant representations of $\mathbf{B}$ are [5]

$$\mathbf{B} = \nabla\psi \times \nabla\theta - \iota\nabla\psi \times \nabla\varphi = J\nabla\varphi + I\nabla\theta + \beta_*\nabla\psi \ . \tag{19}$$

Here θ and φ are the Boozer angle-like magnetic coordinates, $cJ(\psi)/2$ and $cI(\psi)/2$ are the poloidal (external with respect to the magnetic surface) and toroidal electric currents, ι is the rotational transform, and $\Psi = 2\pi\psi$ is the toroidal magnetic flux. The expressions for the Jacobian and for the line element along the field line are given as

$$\sqrt{g} = \frac{1}{\nabla\psi \times \nabla\theta \cdot \nabla\varphi} = \frac{J + \iota I}{B^2}, \qquad \frac{ds}{B} = \sqrt{g}d\varphi, \qquad \theta = \theta_0 + \iota\varphi \ , \tag{20}$$

respectively. With the use of (19), one gets the quantity $|\nabla\psi|k_G$ as

$$|\nabla\psi|k_G = \frac{1}{B^2}(\mathbf{B} \times \nabla B) \cdot \nabla\psi = \frac{1}{J + \iota I}\left(I\frac{\partial B}{\partial\varphi} - J\frac{\partial B}{\partial\theta}\right) . \tag{21}$$

With the Boozer spectrum of B, with the quantities J and I, and with expression (21) for $|\nabla\psi|k_G$, the parameter $\hat{\epsilon}_{\text{eff}}^{3/2}$ in (15) can be computed. So, the quantity $\hat{\epsilon}_{\text{eff}}^{3/2}$ represents that part of $\epsilon_{\text{eff}}^{3/2}$ for which the magnetic field confinement properties are determined through the Boozer spectrum of B and through J and I.

It follows from (15) that for a full account of the effect of plasma geometry on $\epsilon_{\text{eff}}^{3/2}$ the quantity $|\nabla\psi|$ is also essential. For that purpose, one can directly use the Boozer spectrum for $|\nabla\psi|$. Alternatively, one can use the relation with the normal to the magnetic surface, $\mathbf{N}$,

$$\nabla\psi = \frac{1}{\sqrt{g}}\,\mathbf{N}, \quad \mathbf{N} = \frac{\partial\mathbf{r}}{\partial\theta} \times \frac{\partial\mathbf{r}}{\partial\varphi} \ , \tag{22}$$

yielding

$$|\nabla\psi|^2 \ = \ g^{11} \ = \ \frac{1}{g}\left(g_{\theta\theta}g_{\varphi\varphi} - g_{\theta\varphi}^2\right) \ . \tag{23}$$

Here one makes use of the metric tensor g_{ik} and therefore of the Boozer spectra for the coordinates of a magnetic surface. This method is preferred in our numerical realization.

At last, as it follows from (14) and from the equivalence of averages over the magnetic field line and over the volume between two neighboring magnetic surfaces, $\langle|\nabla\psi|\rangle$ is given as

$$\langle|\nabla\psi|\rangle = S\left(\frac{dV}{d\psi}\right)^{-1} = S\left(2\pi\,\frac{dV}{d\Psi}\right)^{-1} \ , \tag{24}$$

with S and V being the magnetic surface area and the volume inside this surface, respectively.

Therefore, the expression for the effective ripple $\hat{\epsilon}_{\rm eff}$ in a Boozer representation can be written as

$$\hat{\epsilon}_{\rm eff}^{3/2} = \frac{\pi R^2}{8\sqrt{2}} \lim_{L_\varphi \to \infty} \left(\int\limits_0^{L_\varphi} \frac{d\varphi}{B^2} \right)^{-1} \int\limits_{B_{min}^{(abs)}/B_0}^{B_{max}^{(abs)}/B_0} {\rm d}b' \sum_{j=1}^{j_{max}} \frac{\hat{H}_{jB}^2}{\hat{I}_{jB}} \ , \tag{25}$$

and the additional integrals over φ (along the magnetic field line)

$$\hat{H}_{jB} = \frac{1}{b'} \int\limits_{\varphi_j^{(min)}}^{\varphi_j^{(max)}} \frac{{\rm d}\varphi}{B^2} \sqrt{b' - \frac{B}{B_0}} \left(4\frac{B_0}{B} - \frac{1}{b'} \right) |\nabla\psi| k_G, \quad \hat{I}_{jB} = \int\limits_{\varphi_j^{(min)}}^{\varphi_j^{(max)}} \frac{{\rm d}\varphi}{B^2} \sqrt{1 - \frac{B}{B_0 b'}} \ , \tag{26}$$

with $|\nabla\psi| k_G$ taken from (21). To compute $\epsilon_{\rm eff}$, one can use

$$\langle |\nabla\psi| \rangle = \lim_{L_\varphi \to \infty} \left(\int\limits_0^{L_\varphi} \frac{d\varphi}{B^2} \right)^{-1} \int\limits_0^{L_\varphi} \frac{d\varphi}{B^2} \sqrt{g^{11}} \ , \tag{27}$$

with g^{11} taken from (23).

Mono-energetic diffusion coefficient

Another approach to compute an effective ripple is a Monte-Carlo computation of the $1/\nu$ neoclassical diffusion coefficients. These computations are often associated with the so called mono-energetic diffusion coefficient $D(E)$. According to the definition of $D(E)$ given by Boozer [6], $D_B(E)$, the average flux is given as

$$\Gamma_t = S F_n = -\int \frac{d\theta_0 d\varphi}{\mathbf{B} \cdot \nabla\varphi} \int\limits_0^\infty D_B(E) \frac{\partial f_M}{\partial\psi} 4\pi v^2 dv = -\frac{dV}{d\psi} \int\limits_0^\infty D_B(E) \frac{\partial f_M}{\partial\psi} 4\pi v^2 dv, \tag{28}$$

where $E = mv^2/2$ is the particle energy (see Eq.(14) in Ref. [6]). In analogy to this definition, one can define the mono-energetic diffusion coefficient in arbitrary coordinates as

$$F_n = -\int\limits_0^\infty D(E, \psi) \frac{\partial f_M}{\partial r} 4\pi v^2 dv \tag{29}$$

with the formal radius r defined in (14). Transforming the integration over $z = mv^2/(2T)$ in (13) back to the integration over v, one obtains

$$F_n = -\frac{\sqrt{2}}{9\pi} \frac{v_T^2 \rho_L^2}{\nu R^2} \epsilon_{\rm eff}^{3/2} \int\limits_0^\infty \frac{1}{A(E/T)} \left(\frac{E}{T} \right)^2 \frac{\partial f_M}{\partial r} 4\pi v^2 dv \ . \tag{30}$$

A comparison with (29) gives for the $1/\nu$ regime

$$D(E, \psi) = \frac{\sqrt{2}}{9\pi} \frac{v^2 \rho^2 \epsilon_{\rm eff}^{3/2}}{\nu A(E/T) R^2}, \tag{31}$$

where $v = \sqrt{2E/m}$ and $\rho = mcv/(eB_0)$. The pitch angle diffusion coefficient is linked to $\nu_\perp$ from the NRL Formulary according to $\nu A(E/T) = \nu_\perp/2$.

Peculiarity of Boozer diffusion coefficients

It is well known, that the Hamiltonian of the guiding center motion with constant magnetic moment shows that the corresponding particle orbits are completely determined by the structure of B. Therefore, in Boozer coordinates, the guiding center equations do not contain other metric elements besides $\sqrt{g}$. Evidently, the quantity $\nabla\psi$ does not enter these equations. However, when calculating the particle flux across a magnetic surface, the geometry of this surface must be taken into account. This leads to the appearance of $\nabla\psi$ in the results in Boozer coordinates. In the work of Boozer [6], the role of the magnetic surface geometry (function ψ) is eliminated from the Boozer diffusion coefficient and transferred to the total flux expression (28). From (28) and (29), taking into account (14) and (24) follows

$$D_B(E) = D(E,\psi)\langle|\nabla\psi|\rangle^2 \, . \tag{32}$$

So, the quantity $\epsilon_{\text{eff}}^{3/2}$ in (15) corresponds to $D(E,\psi)$, whereas the quantity $\hat{\epsilon}_{\text{eff}}^{3/2}$ corresponds to the Boozer diffusion coefficient $D_B(E)$.

4. Parallel current density

The leading order term in the expansion of the distribution function, f_{-1} (see Eq. 7), does not contribute to the parallel plasma current due to its symmetry over σ. Therefore, the current is determined by the next order corrections to the distribution function, g_0 and f_0,

$$\begin{aligned} j_\parallel &= \frac{\pi e B}{m}\sum_{\sigma=\pm 1}\sigma\int\limits_{e\Phi}^{\infty}\mathrm{d}w\int\limits_0^{v^2/B}\mathrm{d}J_\perp \tilde{f} \qquad (33)\\ &= \frac{\pi e B}{m}\sum_{\sigma=\pm 1}\sigma\int\limits_{e\Phi}^{\infty}\mathrm{d}w\int\limits_0^{v^2/B}\mathrm{d}J_\perp\left(g_0 - J_\perp\frac{\partial f_0}{\partial J_\perp}\right). \end{aligned}$$

Here the continuity of f_0 at the trapped-passing boundary was used for integration by parts. Since g_0 is given by (8) as a derivative over $J_\perp$, the contribution to the integral comes from the lower limit $J_\perp = 0$. Due to the current closure condition (2), $H(J_\perp) = 4v^3 Y_{PS}(L)\omega_c/B = 0$ for $J_\perp = 0$. Using the fact that due to Liouville's theorem H exponentially decreases with $L \to \infty$, one can neglect this quantity in (9). Thus one gets

$$\begin{aligned} \frac{j_\parallel}{B} &= -\frac{2\pi e}{m}\int\limits_{e\Phi}^{\infty}\mathrm{d}w\frac{\partial f_M}{\partial\psi}\left(\int\limits_{s_{\min}}^{s}\mathrm{d}s'\frac{4v^3|\nabla\psi|k_G}{3B\omega_c}\right.\\ &+ \left.\int\limits_0^{J_{\perp min}^{(abs)}}\mathrm{d}J_\perp\lim_{L\to\infty}\frac{J_\perp}{I_L}\int\limits_{s_{\min}}^{L}\mathrm{d}s'\frac{|v_\parallel|}{B}\frac{\partial}{\partial J_\perp}\int\limits_{s_{\min}}^{s'}\mathrm{d}s''\frac{V^\psi}{|v_\parallel|}\right). \end{aligned} \tag{34}$$

With the definition of the plasma pressure p

$$p = p(\psi) = \frac{4\pi}{3} \int\limits_{e\Phi}^{\infty} \mathrm{d}w v^3 f_M, \tag{35}$$

and integration over w, one obtains the current density in the following form,

$$\frac{j_\parallel}{B} = -c\,\lambda_\parallel \frac{1}{B_0^2}\left(\frac{dp}{dr} + en\frac{d\Phi}{dr}\right), \qquad \lambda_\parallel = \lambda_{PS}(s) + \lambda_b, \tag{36}$$

where c is the speed of light, n is the plasma density and B_0 is the reference magnetic field. The radial derivative of the pressure is given as

$$\frac{dp}{dr} \equiv \frac{dp}{d\psi}\langle|\nabla\psi|\rangle \ . \tag{37}$$

Here, the dimensionless quantities λ_{PS} and λ_b in (36) which characterize the magnetic field geometry are given in Boozer coordinates

$$\lambda_{PS}(\varphi) = \frac{2}{\langle|\nabla\psi|\rangle} \int\limits_0^{\varphi} \mathrm{d}\varphi' \left(I\frac{\partial\hat{B}}{\partial\varphi'} - J\frac{\partial\hat{B}}{\partial\theta'}\right)/\hat{B}^3 \ , \tag{38}$$

$$\lambda_b = \frac{3}{8\langle|\nabla\psi|\rangle} \lim_{L_\varphi\to\infty} \int\limits_0^1 \mathrm{d}y y^2 \frac{K_b(y)}{L_b(y)} \ , \tag{39}$$

$$K_b(y) = \int\limits_0^{L_\varphi} \mathrm{d}\varphi' \frac{\left(1-y\hat{B}\right)^{1/2}}{\hat{B}^2} \int\limits_0^{\varphi'} \mathrm{d}\varphi'' \left(I\frac{\partial\hat{B}}{\partial\varphi''} - J\frac{\partial\hat{B}}{\partial\theta''}\right)/\left(1-y\hat{B}\right)^{3/2} \ , \tag{40}$$

$$L_b(y) = \int\limits_0^{L_\varphi} \mathrm{d}\varphi' \frac{\left(1-y\hat{B}\right)^{1/2}}{\hat{B}^2} \tag{41}$$

Here the reference magnetic field is $B_0 = B_{max}^{abs}$ and $\hat{B} = B/B_0$. The integration along φ has to be started at the location of this global maximum of B on the flux surface. A representation of λ_{PS} and λ_b in real space coordinates with an arbitrary starting point of integration is given in Ref. [7]. The integration variable y is defined as $y = J_\perp B_0/v^2$ where $y = 1$ marks the trapped-passing boundary. The integration along the magnetic field line can be performed simultaneously with the $\nabla\psi$ calculation. This is actually done by solving a system of linear differential equations when integrating along the field line (see, e.g., [1]).

Contrary to λ_b, the quantity λ_{PS} is a function of φ (or s). One can show that the varying part of λ_{PS} which corresponds to the Pfirsch-Schlüter current, is the same as it is obtained from ideal MHD equilibrium equations. However, those equations do not restrict the constant part of the parallel current. The missing constant part of $j_\parallel/B$ is given by an average value of $j_\parallel/B$. Two definitions of this average are commonly used. The first one, more suitable for equilibrium studies, corresponds to the toroidal current density averaged over the area between two close magnetic surfaces. It is obtained from (36) if $\lambda_\|$ is replaced with λ_{b1},

$$\lambda_{b1} = \frac{\langle \lambda_\| B^\varphi \rangle}{\langle B^\varphi \rangle} = \frac{\langle \lambda_{PS} B^\varphi \rangle}{\langle B^\varphi \rangle} + \lambda_b, \tag{42}$$

where $B^\varphi = \mathbf{B} \cdot \nabla\varphi$ is the toroidal contravariant component of the magnetic field and with surface averages defined in (12). The second definition used in [2] corresponds to the case when the average parallel current vanishes completely in the Pfirsch-Schlüter regime,

$$\lambda_{b2} = \frac{\langle \lambda_\| B^2 \rangle}{\langle B^2 \rangle} = \frac{\langle \lambda_{PS} B^2 \rangle}{\langle B^2 \rangle} + \lambda_b. \tag{43}$$

5. Plateau regime of neoclassical transport

It is well known that for sufficiently small collision frequencies ν, particles with small $v_\|$ give the main contribution to the neoclassical transport as long as the radial electric field is not to high. On the other hand, if ν is still large enough so that effective particle trapping does not occur, the plateau regime of neoclassical transport is realized (see, e.g. [8]). Theoretically this regime was extensively studied either within kinetic theory for rather simple stellarator magnetic field models (e.g. [8,9]) or within fluid theory in magnetic coordinates for more general magnetic field models (e.g. [10]). For solving the drift kinetic equation within this regime, it is convenient to consider the distribution function $f = f_M + \tilde{f}$ as a function of $\mathbf{r}$, w and $v_\|$ (w being the particle energy). With this, the drift kinetic equation becomes

$$v_\| \mathbf{h} \cdot \nabla \tilde{f} + \mathbf{V} \cdot \nabla f_M = St(\tilde{f}) \ , \tag{44}$$

where $f_M = f_M(\psi, w)$ is the Maxwellian distribution function, ψ is the magnetic surface label, $\mathbf{h} = \mathbf{B}/B$, and $\mathbf{V}$ is the drift velocity. In (44), the term $-(\partial \tilde{f}/\partial v_\|) J_\perp \mathbf{h} \cdot \nabla B/2$ ($J_\perp = v^2/B$, v velocity module) has been omitted. This defines the lower limit for $\nu > v_\| \mathbf{h} \cdot \nabla B/B$ in accordance with the usual limit for the plateau regime [8]. In the general case, the transport coefficients in the intermediate region of particle collision frequencies are monotonic functions of the collision frequency (see, e.g. [11]). However, for a sufficiently large aspect ratio of the torus it is known that the neoclassical transport coefficients in this regime practically do not depend on the collision frequency. Theoretically it was shown earlier that the plateau transport coefficients practically do not depend on details of the collision operator and do not differ for various forms of this operator [8,9].

Therefore, the simplest form of the collision operator in the so called τ-approximation was chosen [8]. For this approximation of $St(\tilde{f})$, the drift kinetic equation has the following form,

$$v_\| \mathbf{h} \cdot \nabla \tilde{f} + \mathbf{V} \cdot \nabla \psi \frac{\partial f_M}{\partial \psi} = -\nu \tilde{f} \ . \tag{45}$$

This equation can be written as

$$\frac{\partial \tilde{f}}{\partial s} + p\tilde{f} = -\frac{v^2 B_0}{2 v_\| \omega_{c0}} \left(1 + \lambda^2\right) \frac{|\nabla\psi| k_G}{B} \frac{\partial f_M}{\partial \psi} \ , \tag{46}$$

with $p = \nu/v_\parallel$, and $\lambda = v_\parallel/v$. Computing the single-valued solution of (46) yields the final result which can be presented in the standard form for the plateau regime

$$F_n = -\frac{\sqrt{\pi}\rho_L^2 v_T}{8R\iota\!\!\!-}\Lambda_p \int\limits_0^\infty dz\; z^2 e^{-z}\frac{n}{f_M}\frac{\partial f_M}{\partial r}\;, \tag{47}$$

$$\Lambda_p \;=\; \frac{2RB_0^2\iota\!\!\!-}{\pi\langle|\nabla\psi|\rangle^2}\lim_{L_\varphi\to\infty}\left(\int\limits_0^{L_\varphi}\frac{d\varphi}{B^2}\right)^{-1}\int\limits_{-1}^{1} d\lambda\frac{(1+\lambda^2)^2}{|\lambda|}Z(L_\varphi)\;, \tag{48}$$

where $\rho_L = mcv_T/(eB_0)$ is the mean Larmor radius, and $z = mv^2/(2T)$. Again, the expression for the energy flux density differs from (47) by a factor zT under the integral. The integration over λ shows to be insensitive to the lower and upper limits (-1 and 1) for values of ν which are not too large.

The quantity $Z(L_\varphi)$ has to be computed by integrating the following differential equations over the interval $0 \div L_\varphi$ (for $v_\parallel < 0$) or $0 \div -L_\varphi$ (for $v_\parallel > 0$):

$$\frac{dZ}{d\varphi} \;=\; \frac{|\nabla\psi|k_G}{B^2}\,Y\;, \qquad (Z(0)=0)\;, \tag{49}$$

$$\frac{dY}{d\varphi} \;=\; \sqrt{g}B\left(\frac{|\nabla\psi|k_G}{B^2} + p\,Y\right)\;, \qquad (Y(0)=0)\;, \tag{50}$$

and for $v_\parallel \to 0$

$$\frac{dZ}{d\varphi} = \frac{|v_\parallel|}{\nu}\frac{(|\nabla\psi|k_G)^2}{B^4}, \qquad (Z(0)=0)\;. \tag{51}$$

For an axially symmetric tokamak with large aspect ratio Λ_p equals unity. The deviation of Λ_p from unity gives the effect of the magnetic field geometry on the transport coefficients.

6. Summary

Minimization of neoclassical transport in the so called $1/\nu$ regime is one of the key issues in stellarator optimization. Therefore, our transport analysis was applied to existing configurations (W7-AS, LHD, CHS, HSX, U3-M), to a configuration being build (W7-X), as well as to a variety of candidate configurations in international optimization studies - NCSX, ORNL, CHS-qa - and to the joint effort in stellarator optimization of IPP Greifswald, Russian Research Center “Kurchatov Institute” and IPP at the National Science Center “Kharkov Institute of Physics and Technology”. Within all this efforts, a clear definition of an effective ripple has been introduced which on one hand allows to compare various configurations from the point of view of their transport properties and on the other hand gives a clear link to other definitions of the effective ripple specific to magnetic coordinates.

Here, no numerical results are given. At the moment, an extensive benchmarking effort is under way for $1/\nu$ transport [12] covering a wide variety of configurations. This benchmarking effort will be extended to plateau and parallel current calculations. In addition, different CHS configurations - standard, drift-orbit optimized, quasi-axisymmetric - have been analyzed using a real space version representation [13]. Preliminary results on parallel currents have been published in Refs. [14,7,15], and on plateau transport in Ref. [16]. A version of the code NEO utilizing Boozer coordinates is freely distributed among research laboratories.

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