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3D STELLARATOR CODES

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Abstract

Large computer codes have been used to develop compact stellarators with modular coils that are promising candidates for a magnetic fusion reactor. The mathematics of plasma confinement raises significant issues about these numerical calculations. Convergence studies have been performed to assess the best configurations, and comparisons with recent data from the LHD experiment validate the theory.

1. Introduction

The NSTAB equilibrium code is a computer implementation of the variational principle of ideal magnetohydrodynamics. Solutions of the magnetostatics equations are found by minimizing the potential energy

$$E = \int\int\int \left[B^2/2 - p\right] dV$$

in a coordinate system compatible with toroidal geometry in three dimensions. An accurate finite difference scheme is employed in the radial direction, and dependence on the poloidal and toroidal angles is handled by the spectral method. It is assumed that there are nested toroidal flux surfaces, and the differential equations are written in a conservation form that captures islands and current sheets. The resolution is so good that questions of stability can be settled by a mountain pass theorem asserting that when more than one solution of the problem can be found then an unstable equilibrium must exist corresponding to a saddle point in the energy landscape. For stellarators bifurcated equilibria are calculated whose magnetic surfaces have Poincaré sections displaying the structure of the most unstable modes.

The TRAN Monte Carlo code employs a split time algorithm to calculate the confinement time of test particles by alternately tracking guiding center orbits and applying a random walk that represents collisions. The magnetic field B and the flow field U of the plasma in a background obtained using NSTAB are held fixed during iterations that impose quasineutrality. Conservation of momentum might be enforced by the selection of U, but because it is not thermal U has little effect on the collision operator. On the other hand, anomalous transport can be modeled by iterating on small variations of the electric potential within the magnetic surfaces to achieve quasineutrality between the distributions of ions and electrons. The method simulates complicated transport in stellarators remarkably well, and numerical results have been obtained for the LHD experiment in Japan that are in excellent agreement with recent observations of the energy confinement time at high temperatures.

2. Convergence studies

Calculation of toroidal equilibrium of a plasma without two-dimensional symmetry is a problem in mathematics that is not well posed. In numerical work this leads us to construct weak, discontinuous solutions of discrete equations that are expressed in conservation form, but even the best methods only converge in an asymptotic sense. Enough spectral terms must be included to eliminate significant truncation error, but not so many that the results become meaningless. We shall present convergence studies for the NSTAB code that explain how this can be accomplished.

As an example for convergence studies we have selected a compact stellarator called the MHH2 that has two field periods, plasma aspect ratio three, excellent quasiaxial symmetry, and an average β limit near 5%. Applying the method of steepest descent to the variational principle, we test for stability by examining runs of the NSTAB code in which some dangerous mode has been triggered by introducing temporarily an appropriate forcing term. A run predicts stability if the mode decays during further iterations, but if it grows then the equilibrium is unstable. A more convincing conclusion can be drawn from the mountain pass theorem if the iterations converge to a bifurcated solution whose stellarator symmetry is visibly broken by magnetic surfaces that exhibit the structure of the suspicious mode. The numerical results depend on the maximum degree N of factors in each of two angular coordinates that specify the spectral terms included in the computation. The purpose of our convergence study is to decide whether the prediction about stability has a meaningful limit as N increases.

For the example of the MHH2 we have performed equilibrium and stability runs of the NSTAB code with between 14 and 28 mesh intervals in the radial flux coordinate s and with a largest degree N of the spectral terms in each of two angular flux coordinates ranging as high as $N = 48$. Because of the nested surface hypothesis, islands are captured better on crude grids, and the calculations are relatively insensitive to the radial mesh size because the finite difference scheme in s has an especially accurate conservation form. The preconditioned iterative scheme in the code was tuned to emphasize steepest descent of the energy over speed in decay of the residuals so that the test of stability would become more reliable.

In Table 1 we compare the degree N of the spectral calculations with the corresponding estimate of an average β limit based on the mountain pass theorem. If the degree is low the method does not provide meaningful results about stability. However, for $N = 24$ one obtains efficiently answers that are of sufficient accuracy to optimize the design of a quasisymmetric stellarator like the MHH2. The results do not change significantly for $N > 32$, which is a good value to choose from the point of view of asymptotic convergence. At $N = 48$ a stage is reached where the numerical method may fail in long runs because of the singular behavior of the solution.

Table 2 shows how the convergence of the iterative scheme employed in the NSTAB code depends on the degree N of the spectral terms that are used. For crude grids it is possible to reduce the residuals to the level of rounding error in the computer.

However, as N increases the effectiveness of the scheme deteriorates, and the method may only converge in the asymptotic sense that at first the errors become smaller, but later they increase without limit. The numerical data establish that the most reliable results are calculated for degrees in the range $20 \leq N \leq 32$. Since in many cases the residuals decrease indefinitely, the system of discrete equations implemented in the NSTAB code does seem in general to have a solution. It is more difficult to interpret numerical results for methods in which that is not the case.

3. Comparison with the LHD experiment

An important test of numerical methods is comparison with experiment. Recent communications from Japan about the LHD stellarator there state that average values of β between 2.5% and 3.5% have been observed. The success of the LHD experiment in achieving such good performance shows that stellarators have desirable physical properties despite the nonexistence of smooth solutions of the system of partial differential equations for magnetohydrodynamic equilibrium. Comparison with the observations increases our confidence in the mathematical model of a weak solution approximated asymptotically by numerical computations. An NSTAB calculation simulating the experiment was performed where the magnetic axis was assumed to have major radius $R = 3.6$ m and the radial mesh size was 1/27. When the pressure profile was adjusted to optimize our estimate of the β limit, the numerical work became consistent with the measurements, but in some respects the LHD experiment outperformed theoretical predictions. Perhaps it is best to treat the three-dimensional (3D) computer codes not as a refined mathematical tool, but as a sophisticated simulation to be adjusted like a scaling law to fit the observations.

Predictions of nonlinear stability found by running the NSTAB code agree well with experimental observations in both stellarators and tokamaks. However, the same is not true of local theories such as the ballooning criterion evaluated by solving an ordinary differential equation along magnetic lines. We have performed numerical computations showing that that method gives results for the β limit of the LHD significantly lower than the measurements we have cited. Convergence studies demonstrate that a more reliable prediction of β limits for ideal magnetohydrodynamic modes in both the LHD and the MHH2 is obtained from applications of the NSTAB code that determine bifurcated equilibria whose magnetic surfaces have a localized ballooning structure. The techniques we have developed to perform these calculations effectively on fine grids using spectral terms of high degree have allowed us to optimize the design of the MHH2 stellarator so that it has acquired good physical properties combined with 3D geometry that is not too complicated.

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N	16	20	24	32	48
β	0.060	0.050	0.045	0.040	0.039

Table 1: Relationship between the highest degree N of the spectral terms used in a run of NSTAB and the resulting critical value of β that is calculated.

N	16	20	24	32	48
cycle	1×10^6	5×10^5	1×10^5	5×10^4	1×10^4
error	10^{-9}	10^{-7}	10^{-5}	10^{-4}	10^{-3}

Table 2: Relationship between the highest degree N of the spectral terms used in runs of NSTAB, the number of iterations that were run successfully for this value of N, and the amount that the residuals decayed. In the finite difference scheme 14 intervals of the radial coordinate s were employed.

References

1. P. Garabedian and L. Ku, Quasiaxially symmetric stellarators with three field periods, Phys. Plasmas **6** (1999) 645-648.

2. P. Garabedian and M. Taylor, Tokamak transport driven by quasineutrality and helical asymmetry, Nucl. Fusion **32** (1992) 265-270.

3. O. Motojima, private communication (March 2001).

4. M. Taylor, A high performance spectral code for nonlinear MHD stability, J. Comp. Phys. **110** (1994) 407-418.

5. H. Weitzner, Conditions for the existence of steady state stellarators, Phys. Plasmas **5** (1998) 417-432.